مقاسات جزئية اولية من النمط – R

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R-Prime Submodules

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Abstract. R is a commutative ring with unity. K is a sub-module of a non- zero left R-module D, In this paper, first we define the notion of residual of a Sub-module in D which is an ideal in a ring R and then we obtain some related results. In particular ,we state and prove some proposition. We need in this paper some concepts as multiplication module , prime sub-module and an injective envelope . Finally we will introduce a new concept "R −prime sub-module ". In this paper concepts of R −prime sub-modules and some other notions will be studied, as well as the relationship between properties of R −prime sub-modules and other sub-modules . we will be studied image of R −prime sub-module . Also the relationship between the properties of prime sub-modules and other sub-modules.

Keywords: Semiprime , sub-module, Prime sub-module, J −Semiprime sub-module, Prime ideal, Regular and multiplication**.**

1 Introduction

The whole rings are commutative with individuality. The whole modules are individual. Many studies and searches are documented about the prime sub-modules by several researchers who are interested with the commutative algebra topic and among them as J. Dauns[1]. In this paper we need the following definition which comes in [2]. K is $\boldsymbol{J}$-semiprime if whenever $r^n \boldsymbol{x} \in$ K + $\boldsymbol{J}$(D); $\boldsymbol{r} \in \boldsymbol{R}$, $\boldsymbol{x} \in$ D, n $\in \boldsymbol{Z}$+, implies that $\boldsymbol{rx} \in$ K**.** The definition comes in [1] as following: A convenient sub-module K is a prime sub-module whenever $r\,d \in K,\ r \in R$ and $d \in D$, then either $d \in K$ or $r \in [K{:}D] = \{r \in R : rD \subseteq K\}$, that they are generalized of prime ideals, which got big importance at last year. In this study we introduce the following R −prime sub-module whenever $r\,d \in K + J(D), r \in R, d \in D$, up to $d \in K$ or $r \in [K{:}D]$; $J(D)$ is the Jacobson radical of (D) and study a new kind of an ideal which is said to be R −prime; $I \neq R$ whenever $s.t \in J(R) + I$, then either $s \in I$ or $t \in I$; $J(R)$ is the Jacobson radical of R .

2 Preliminaries

A commutative ring having individuality and D be a non-zero individual left –module. K is named R −prime sub-module whenever $r\,d \in K + J(D), r \in R, d \in D$, up to $d \in K$ or $r \in [K{:}D]$; $J(D)$ is the Jacobson radical of (D). Remember that $E_n(K)$ is said to be an injective envelope of a sub-module K of the R −module (D), it is defined as follows : $E_n(K) = \{x = r.t : r \in R , t \in D\ such\ that\ r^n , n \in Z^+\}$. It is clear that $\subseteq E_n(K)$, [3]. Recall that if (D) is a $R - module$, thus, (D) is known as a multiplication module when each sub-module K of (D), and a perfect I of a ring R is exist, $K = ID$ [5]. Let P be a proper sub-module of a multiplication module M. Then P is prime if and only if UV ⊆ P ⇒ U ⊆ P or V ⊆ P[6]. Let P be a proper sub-module of M. Then P is prime if and only if m·m ⊆ P ⇒ m ∈ P or m ∈ P[6].

3 R −prime Submodule

Bearing in mind that a convenient sub-module K of $R-$ module D is named a prime sub-module whenever $r\,d \in K,\ r \in R,\ d \in D$, then either $d \in K$ or $r \in [K{:}D] = \{r \in R : rD \subseteq K\}$ [2].

The followings are introduced:

Definition (3.1):

$K \neq D$ is called R −prime sub-module whenever $r\,d \in J(D) + K, r \in R, d \in D,$ implies that $d \in K$ or $\in [K{:}D]$.

Remark (3.2):

For all prime sub-modules of R −module D is the R −prime sub-module, and the opposite is not generally correct. If $D = Z_4 \oplus Z_2$ is $Z - module\ D$, $K = \bar{0} + Z_2 = \{(\bar{0},\bar{0}),(\bar{0},1)\}$ is the sub-module.
Now, $3(\bar{2},\bar{0}) \in J(D) + K$, $J(D) = \{(\bar{0},\bar{0}),(\bar{2},\bar{0})\}$, then $3 \notin [K{:}D] = \{0\}$, $(\bar{2},\bar{0}) \notin K$. Which implies that K is not R −prime sub-module, while K is a prime sub-module.

Example(3.3):

The module $Z_P\infty$ over Z has no R −prime sub-module, where p is the prime number, since $Z_P\infty$ has no prime submodule.

Proposition (3.4):

For each maximum sub-module of R −module D is a R −prime sub-module. In special case, the module Q over Z the only R −prime sub-module is 0.

Proof:

$rt \in J(D) + K, r \in R, t \in D$. Since K is the maximum sub-module, then $[K{:}D]$ is the ideal prime, and $J(D) \subseteq K$. Hence, $rt \in K$, but K is the prime sub-module, thus $t \in K$ or $r \in [K{:}D]$.

Example(3.5):

$K = \langle\bar{4}\rangle$ as Z_8 sub-module, then K is not R −prime sub-module of Z_8 over Z.

Proposition (3.6):

Every J −semiprime sub-module is R −prime sub-module.

Proof:

Allow $rt \in J(D) + K,\ t \in M,\ r^k \in K + J(D),\ k \in z^+,\ r \in R,$ but K is J −semiprime submodule, then $rt \in K$, then $r \in [K{:}D]$.

Proposition (3.7):

When D is a regular module, thus each prime sub-module is R −prime submodule.
The ideal R −prime of a ring R can be defined.

Definition (3.8):

An ideal I is said to be R −prime; $I \neq R$ whenever $s.t \in J(R) + I$, then either $s \in I$ or $t \in I$; $J(R)$ is the Jacobson radical of R .

Proposition (3.9):

Let K is R −prime (D) sub-module, then $[K{:}D]$ is the ideal R −prime of R.

Proof:

Let $s.t \in [K:D] + J(R)$, for each $s.t \in R$. It is wanted to show either $s \in [K:M]$ or $t \in [K:M]$. If $m \in D$, $(s.t).m = r.m + j.m$, $r \in [K:D]$, $j \in J(R)$, then $(s.t).m \in J(D) + K$, but K is the (D) R −prime sub-module, thus either $m \in K$ or $s.t \in [K:D]$, but $[K:D]$ is the ideal prime, hence $t \in [K:D]$ or $s \in [K:D]$. Which implies that $[K:D]$ is the ideal R −prime of R. In general, the opposite isn't correct, for instant, $K = \langle(4,0)\rangle$ is a sub-module of $M = Z \oplus Z$ over a ring Z, $[K:D] = 0$ is the ideal R −prime of Z, but K isn't R −prime sub-module, $2(2,0) \in J(M) + K$, and neither $2 \in [K:M]$ nor $(2.0) \in K$.

Proposition (3.10):

When K being the proper sub-module of R −module (D), thus K is the R −prime sub-module, when $K = E_n(J(D) + K)$.
Proof:
K is the R −prime sub-module of D). It is enough to prove $E_n(K + J(D)) \subseteq K$.
Let $x = l.s \in E_n(J(D) + K), l \in R, s \in D$, and there exist $n \in Z^+$ such that $l^n.s \in K + J(D)$, but K is R −prime sub-module, then $s \in K$ or $l^n \in [K:D]$. Because $[K:D]$ is an ideal prime, then $l \in [K:D]$. In both cases $x = l.s \in K$. That includes $K = E_n(J(D) + K)$. For converse, suppose that $x = r.s \in K + J(D) \subseteq E_n(J(D) + K) = K, r \in R, s \in D$, then $r.s \in K$, therefore $rD \subseteq K$, for all $s \in D$, hence $r \in [K:D]$. This up to K is R −prime sub-module of D.
Compare the following with Proposition (2.1):[4]

Proposition (3.11):

If $\psi: D \longrightarrow D'$ is an epimorphism; D and D′ are two modules over a ring R with $ker\,\psi \subseteq K$ and $ker\,\psi \ll D$, then:

(1) If K is the R −prime (sub-module) of (D), thus ψ(K) is R −prime sub-module of (D′).

(2) If K′ being the R −prime (sub-module) of (D′), thus ψ^{-1} (K) is R −prime sub-module of (D).

Proof (1): ψ(K) ≠ D′. One gets a contradiction if (K) ≠ D, since ψ(v) ∈ ψ(K) for the whole $v \in D$, then $\exists\, k \in K$ such that ψ(v) = ψ(k), then $v - k \in ker\,\psi \subseteq K$, thus $D = K$, which is the contradiction,since $K \neq D$.
Now, $l.v' \in J(D') + \psi(K)$, $l \in R, v' \in D'$. It is wanted to depict that ψ(K) is the R −prime sub-module of D′. $\exists v \in D$ such that $\psi(v) = v'$, thus $l.v' = l\,\psi(v) = \psi(l\,v) \in J(D') + \psi(K) = \psi(J(D)) + \psi(K)$, then $\psi(l\,v) = \psi(s) + \psi(y)$; $y \in K, s \in J(D)$, $l.v - (y + s) \in ker\,\psi \subseteq K$ $l.m \in J(D) + K$, but K is *the* R −prime sub-module, then either $l \in [K:D]$ or $v \in K$.
Hence $v' = \psi(v) \in \psi(K)$ or $l \in [\psi(K):D']$. Which implies that $\psi(K)$ is R −prime submodule of D'.
Proof (2): $\psi^{-1}(K') \neq D$. One gets a contradiction if $\psi^{-1}(K') = D$. Since $\psi^{-1}(K') = D$, then $K' = \psi(D) = D'$, this is a contradiction. Now, let $l.v \in J(D) + \psi^{-1}(K'), l \in R, v \in D$. It is wanted to reveal that $\psi^{-1}(K')$ is the R −prime sub-module of (D). $l.v \in J(D) + \psi^{-1}(K')$, then $l.\psi(v) \in J(D') + K'$, but K' is the R −prime sub-module of (D'), thus either $l \in [K':D']$ or $\psi(v) \in K'$. Hence $m \in \psi^{-1}(K')$ or $l \in [\psi^{-1}(K'):D]$. This implies that $\psi^{-1}(K')$ is the R −prime sub-module of (D).

4 Conclusion

By $Theorem(3.16)$ and $corollary$ (3.17) , [6] and same way the next will be proved

Theorem(4.1): When P being the sub-module of a multiplication module, $P \neq M$, thus the next are equivalent:

(1) A sub-module (P) is the R $-$prime.

(2) For each sub-module C_1 and C_2 of D, if $C_1 . C_2 \subseteq P + J(D)$, then either $C_1 \subseteq P + J(D)$ or $C_2 \subseteq P + J(D)$.

(3) For every $c_1, c_2 \in D$, if $c_1 . c_2 \in P + J(D)$, then either $c_1 \in P + J(D)$ or $c_2 \in P + J(D)$.

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